NONRESONANCE PARAMETRIC INTERACTIONS OF SURFACE WAVES IN ISOTROPIC SOLID BODIES

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A solution in parametric approximation is given of the nonlinear interaction problem of Rayleigh surface waves propagating in a solid body with given elastic fields. Shortened equations that govern the modulation effect of the surface waves, and also expressions for the modulation index in terms of the third-order elasticity constant, are obtained.

1. The generation of higher harmonics has only been analyzed in literature out of the many nonlinear effects which take place when surface waves are propagated in solid bodies. This, however, does not exhaust all the nonlinear effects; it is of some interest to study interactions between several surface waves as well as the interaction between the surface waves with the interior elastic fields of the solid medium. In the present article a theoretical analysis is carried out of the parametric interactions between Rayleigh surface waves and the volume elastic fields which satisfy boundary conditions.

The analysis can be carried out by using asymptotic methods as its basis. To this end one has to solve a system of nonlinear wave equations with appropriate boundary conditions.

The problem is formulated as follows: let on the solid body–vacuum interface a Rayleigh wave be propagated in the direction of the x axis (the usual Cartesian coordinate system is used) which is uniform in y, but nonuniform in z (the normal vector to the solid-body surface is directed along the z axis). An arbitrary external modulating field (satisfying suitable boundary conditions) which changes slowly in space and in time when compared with the oscillations of the points in the Rayleigh wave acts upon the solid body. The problem is reduced to the finding of the complex amplitude of the Rayleigh wave. When Lagrange variables are used the system of nonlinear wave equations for this case is given by

$$\begin{aligned}
&\rho_0 \frac{\partial^2 u_1}{\partial t^2} - \left(K + \frac{4}{3}\mu\right)\frac{\partial^2 u_1}{\partial x^2} - \mu\frac{\partial^2 u_1}{\partial z^2} - \left(K + \mu\frac{1}{3}\right)\frac{\partial^2 u_3}{\partial x \partial z} = \frac{\partial T_{11}}{\partial x} + \frac{\partial T_{13}}{\partial z} \\
&\rho_0 \frac{\partial^2 u_3}{\partial t^2} - \left(K + \frac{4}{3}\mu\right)\frac{\partial^2 u_3}{\partial z^2} - \mu\frac{\partial^2 u_3}{\partial x^2} - \left(K + \frac{\mu}{3}\right)\frac{\partial^2 u_1}{\partial x \partial z} = \frac{\partial T_{31}}{\partial x} + \frac{\partial T_{33}}{\partial z}
\end{aligned} \tag{1.1}$$

where K is the uniform-compression modulus, μ is the shear modulus, T_{ik} is the nonlinear part of the stress tensor, u_i are the components of the displacement vector (the subscript 1 corresponds to x, 2 ~ y, 3 ~ z). The boundary conditions which consist of the absence of forces normal to the surface of the solid body are given by [1]

$$\sigma_{ik} n_k = 0 \quad \text{for} \quad z = 0 \tag{1.2}$$

where n_k is the vector normal to the boundary of the solid elastic medium. In this case with normal in the direction of z one has the relations

$$\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0 \tag{1.3}$$

where σ_{ik} is the complete stress tensor.

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By using (1.2), (1.3) and in view of the fact that the wave is uniform in y (that is, $\partial/\partial y = 0$) one can obtain boundary conditions for $z = 0$,

$$\mu\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) = -T_{13}$$
$$\left(K + \frac{4}{3}\mu\right)\frac{\partial u_3}{\partial z} + \left(K - \frac{2}{3}\mu\right)\frac{\partial u_1}{\partial x} = -T_{33} \tag{1.4}$$

The problem is thus reduced to the solving of Eq. (1.1) with the boundary conditions (1.4) for a given modulating field $\mathbf{u}^m$.

Before tackling this nonlinear problem the solution of the linear problem, that is, the solution of the system (1.1) together with (1.4) in the case of a vanishing right-hand side, is written down in the following form:

$$u_1 = \mathrm{Re}\,(a_1 e^{\varkappa_1 z} + a_2 e^{\varkappa_2 z}) e^{i(\omega t - kx)}$$
$$u_3 = \mathrm{Re}\,(i\rho_1 a_1 e^{\varkappa_1 z} + i\rho_2 a_2 e^{\varkappa_2 z}) e^{i(\omega t - kx)} \tag{1.5}$$

where a_1 and a_2 are constant amplitudes and

$$\varkappa_1^2 = k^2 - \frac{\omega^2}{c_l^2}, \quad \varkappa_2^2 = k^2 - \frac{\omega^2}{c_t^2}, \quad \rho_1 = -\frac{\omega^2 - c_l^2 k^2}{\varkappa_1 k c_l^2}, \quad \rho_2 = \frac{k}{\varkappa_2}$$
$$\left(c_l = \left[\frac{(K + {}^4/_3\,\mu)}{\rho_0}\right]^{1/2}, \quad c_t = (\mu/\rho_0)^{1/2}\right)$$

k is the wave number, c_l is the velocity of the longitudinal waves, c_t the velocity of the displacement waves, ρ_0 is the density of the undisturbed medium. For a wave which is uniform in y the conditions of the linear problem are satisfied if

$$a_2 = a_1 S \qquad \left(S = -\frac{2\rho_1\rho_2}{1 + \rho_2^2}\right) \tag{1.6}$$

and the corresponding dispersion equation has a solution of the form

$$\omega = c_t k \xi \tag{1.7}$$

where ξ is a constant value smaller than unity.

2. The solution of nonlinear interaction of a Rayleigh wave and a modulating elastic field is sought in parametric form, namely

$$u_1(x, t) = \mathrm{Re}\,[a_1(x, t)\, e^{\varkappa_1 z} + a_2(x, t)\, e^{\varkappa_2 z}]\; e^{i(\omega t - kx)} + u^m(x, y, z, t) + \mu^* w_1$$
$$u_3(x, t) = \mathrm{Re}\,[i\rho_1 a_1(x, t)\, e^{\varkappa_1 z} + i\rho_2 a_2(x, t)\, e^{\varkappa_2 z}] e^{i(\omega t - kx)} + u^m(x, y, z, t) + \mu^* w_2 \tag{2.1}$$

where $a_1(x, t)$ and $a_2(x, t)$ are slowly varying functions of x and t, μ^* is a dimensionless parameter in the sense of the acoustic Mach number, $(\mu^* = u/\lambda)$; w_i is a residual term which takes into account the fact that the approximate solution (2.1) is not exact. By inserting (2.1) in (1.1) and comparing the coefficients of $e^{i(\omega t - kx)}$ after expansion in Fourier series one can obtain the equations for w_i,

$$(c_l^2 k^2 - \omega^2) w_1 - c_t^2 \frac{\partial^2 w_1}{\partial z^2} + ik(c_l^2 - c_t^2)\frac{\partial w_3}{\partial z} = \left(\alpha_{11}\frac{\partial a_1}{\partial t} + \alpha_{12}\frac{\partial a_1}{\partial x} + \right.$$
$$\left. + \alpha_{13}\frac{\partial a_1}{\partial z} + F_{11}\right) e^{\varkappa_1 z} + \left(\beta_{11}\frac{\partial a_2}{\partial t} + \beta_{12}\frac{\partial a_2}{\partial x} + \beta_{13}\frac{\partial a_2}{\partial z} + F_{12}\right) e^{\varkappa_2 z}$$
$$(c_t^2 k^2 - \omega^2) w_3 - c_l^2 \frac{\partial^2 w_3}{\partial z^2} + ik(c_l^2 - c_t^2)\frac{\partial w_1}{\partial z} = \left(\alpha_{21}\frac{\partial a_1}{\partial t} + \alpha_{22}\frac{\partial a_1}{\partial x} + \right.$$
$$\left. + \alpha_{23}\frac{\partial a_1}{\partial z} + F_{13}\right) e^{\varkappa_1 z} + \left(\beta_{21}\frac{\partial a_2}{\partial t} + \beta_{22}\frac{\partial a_2}{\partial x} + \beta_{23}\frac{\partial a_2}{\partial z} + F_{32}\right) e^{\varkappa_2 z} \tag{2.2}$$
$$(\alpha_{11} = -2i\omega, \ \alpha_{12} = -ik\,[2c_l^2 - \rho_1^2(c_l^2 - c_t^2)], \ \alpha_{13} = k\rho_1(c_l^2 + c_t^2)$$
$$\alpha_{21} = 2\rho_1\omega, \ \alpha_{22} = k\rho_1(c_l^2 + c_t^2), \ \alpha_{23} = ik\,[2\rho_1^2 c_l^2 - (c_l^2 - c_t^2)]$$
$$\beta_{11} = -2i\omega, \quad \beta_{12} = -ik\rho_1(c_l^2 + c_t^2), \ \beta_{13} = \frac{k}{\rho_2}[2c_t^2 + \rho_2^2(c_l^2 - c_t^2)]$$

$$\beta_{21} = 2\rho_2\omega, \; \beta_{22} = -\frac{k}{\rho_2}\,[2\rho_2^2 c_t^{\,2} + (c_l^{\,2} - c_t^{\,2}],\quad \beta_{23} = ik\,(c_l^{\,2} + c_t^{\,2}))$$

where F_{ik} are Fourier coefficients obtained by averaging the right-hand sides of Eqs. (1.1). In the derivations of F_{ik} one makes use of the inequalities

$$\begin{gathered}\left|\frac{\partial^2 a_i}{\partial x_i \partial x_k}\right| \ll k\left|\frac{\partial a_i}{\partial x_k}\right| \ll k^2 |a_i|, \quad \left|\frac{\partial^2 a_i}{\partial t^2}\right| \ll \omega\left|\frac{\partial a_i}{\partial t}\right| \ll \omega^2 |a_i| \\ \left|\frac{\partial^2 u_i^m}{\partial x_i \partial x_k}\right| \ll k\left|\frac{\partial u_i^m}{\partial x_k}\right| \ll k^2 |u_i^m|; \quad \left|\frac{\partial^2 u_i^m}{\partial t^2}\right| \ll \omega\left|\frac{\partial u_i^m}{\partial t}\right| \ll \omega^2 |u_i^m|\end{gathered} \tag{2.3}$$

The coefficients F_{ik} are given by the relations

$$\begin{aligned}
&F_{11}, F_{31} = \langle F_i\rangle \{i=1.3,\ \gamma = \varkappa_1,\ A_1 = a_1,\ A_2=0,\ A_3=i\rho_1 a_1\}\\
&F_{12}, F_{32} = \langle F_i\rangle \{i = 1.3;\ \gamma=\varkappa_2;\ A_1=a_2;\ A_2=0,\ A_3=i\rho_2 a_2\}\\
&\langle F_i\rangle = \left(\mu + \frac{A}{4}\right)\left(-k^2 A_l \frac{\partial u_l^m}{\partial x_i} + \gamma^2 A_l \frac{\partial u_l^m}{\partial x_i} - k^2 A_l \frac{\partial u_i^m}{\partial x_l} + \gamma^2 A_l \frac{\partial u_i^m}{\partial x_l} -\right.\\
&\left. - 2k^2 A_i \frac{\partial u_1^m}{\partial x} + 2\gamma^2 A_i \frac{\partial u_3^m}{\partial z} - 2ik\gamma A_i \frac{\partial u_1^m}{\partial z} - 2ik\gamma A_i \frac{\partial u_3^m}{\partial x}\right) +\\
&+ \left(K + \frac{\mu}{3} + \frac{A}{4} + B\right)\left(-k^2\delta_{1i} A_l \frac{\partial u_i^m}{\partial x} + \gamma^2 \delta_{3i} A_l \frac{\partial u_l^m}{\partial z} - ik\gamma\delta_{1i} A_l \frac{\partial u_l^m}{\partial z} -\right.\\
&- ik\gamma\delta_{3i} A_l \frac{\partial u_l^m}{\partial x} - ik\gamma A_3 \frac{\partial u_i^m}{\partial x} - k^2 A_1 \frac{\partial u_i^m}{\partial x} - ik\gamma A_1 \frac{\partial u_i^m}{\partial z} +\\
&\left.+ \gamma^2 A_3 \frac{\partial u_1^m}{\partial z}\right) + \left(K - \frac{2}{3}\mu + B\right)\left(-k^2 A_i \frac{\partial u_l^m}{\partial x_l} + \gamma^2 A_i \frac{\partial u_l^m}{\partial x_l}\right) +\\
&+ \left(B + \frac{A}{4}\right)\left(-k^2 A_1 \frac{\partial u_1^m}{\partial x_i} + \gamma^2 A_3 \frac{\partial u_3^m}{\partial x_i} + \gamma^2 \delta_{3i} A_l \frac{\partial u_3^m}{\partial x_l} - ik\gamma A_1 \frac{\partial u_3^m}{\partial x_i} -\right.\\
&- k^2 \delta_{1i} A_l \frac{\partial u_1^m}{\partial x_l} - ik\gamma A_3 \frac{\partial u_1^m}{\partial x_i} - ik\gamma\delta_{3i} A_l \frac{\partial u_1^m}{\partial x_l} -\\
&\left.- ik\gamma\delta_{1i} A_l \frac{\partial u_3^m}{\partial x_l}\right) + (B + 2C)\left(-k^2\delta_{1i} A_1 \frac{\partial u_l^m}{\partial x_l} - ik\gamma\delta_{3i} A_1 \frac{\partial u_l^m}{\partial x_l} - ik\gamma\delta_{1i} A_3 \frac{\partial u_l^m}{\partial x_l} + \gamma^2\delta_{3i} A_3 \frac{\partial u_l^m}{\partial x_l}\right)
\end{aligned} \tag{2.4}$$

It follows from (2.2) that the differential operator acting on the left side on W_i is identical with the corresponding differential operator for the linear problem. The function w_i now changes "rapidly" with z, that is, the space scale of w_i with respect to z is of the order of the wavelength λ. The equations for w_1 and w_3 form a quasilinear in z nonautonomous system. The forced solution of this problem can be obtained as a superposition of solutions for separate components of the applied force on the right-hand side of Eqs. (2.2).

It was shown in the theory of asymptotic methods that the approximate solution (2.1) converges to the actual solution if w_i varies with respect to the "rapid" variables just the same as does the solution of the linear problem. For this condition to be satisfied it is required that

$$w_i = w_i^{\circ} e^{\varkappa z} \tag{2.5}$$

By substituting (2.5) into (2.2) an algebraic system for w_i° can be obtained, namely

$$R_1 w_1^{\circ} + p_1 w_2^{\circ} = Q_1, \quad R_2 w_3^{\circ} + p_2 w_1^{\circ} = Q_2 \tag{2.6}$$

The determinant for the system (2.6) vanishes. If nonaccumulation, that is, boundedness of w_i° is required, and if the consistency condition of the system (2.6) is satisfied, the sought shortened equations for slowly varying amplitudes a_1(x, t) and a_2(x, t) can be obtained,

$$\begin{gathered}\frac{\partial a_1}{\partial t} + \frac{c_l^{\,2}}{c_t \xi}\frac{\partial a_1}{\partial x} + i\frac{c_l^{\,2}\rho_1}{c_t \xi}\frac{\partial a_2}{\partial z} + \frac{iF_{11} - \rho_1 F_{31}}{2\omega(1 - \rho_1^2)} = 0 \\ \frac{\partial a_2}{\partial t} + \frac{c_t}{\xi}\frac{\partial a_2}{\partial x} + i\frac{c_t}{\rho_2 \xi}\frac{\partial a_1}{\partial z} + \frac{iF_{12} - \rho_2 F_{32}}{2\omega(1 - \rho_2^2)} = 0\end{gathered} \tag{2.7}$$

The relation (1.7) was taken into account when (2.7) was being derived. It was assumed when deriving (2.7) that the structure of the modulating waves with respect to z remains the same as for the linear problem but the amplitude changes "rapidly" in z.

3. The latter must be taken into account when deriving the averaged boundary conditions. To obtain them one inserts (2.1) into (1.4). Comparing the terms of the order μ^* in the expressions thus derived one can write the system for w_i as

$$
\begin{aligned}
&c_t^2 \frac{\partial w_1}{\partial z} - ikc_t^2 w_3 = \Phi_{13} - i\rho_1 c_t^2 \frac{\partial a_1}{\partial x} - i\rho_2 c_t^2 \frac{\partial a_2}{\partial x} \\
&c_l^2 \frac{\partial w_3}{\partial z} - ik(c_l^2 - 2c_t^2) w_1 = \Phi_{33} - i\rho_1 c_l^2 \frac{\partial a_1}{\partial z} - i\rho_2 c_l^2 \frac{\partial a_2}{\partial z}
\end{aligned} \tag{3.1}
$$

The boundary conditions are written down for the surface z = 0; the latter should be taken into account when differentiating with respect to z in (3.1). The quantities Φ_{ik} in (3.1) are the corresponding Fourier coefficients obtained when averaging the right-hand sides of (1.4) over the rapid variables

$$
\begin{aligned}
&\Phi_{13},\ \Phi_{33} = \langle T_{ik}\rangle \{i = 1.3;\ k = 3,\ A_1 = a_1,\ A_2 = 0,\ A_3 = i\rho_1 a_1, \\
&\gamma = \varkappa_1\} + \langle T_{ik}\rangle \{i = 1.3, k = 3, A_1 = a_2, A_2 = 0, A_3 = i\rho_2 a_2, \gamma = \varkappa_2\} \\
&\langle T_{ik}\rangle = \left(\mu + \frac{A}{4}\right)\left(-ikA_l\delta_{13}\frac{\partial u_l^m}{\partial x_k} - \gamma A_l\delta_{3i}\frac{\partial u_l^m}{\partial x_k} - ikA_l\delta_{1k}\frac{\partial u_l^m}{\partial x_i} + \right. \\
&+ \gamma A_l\delta_{3k}\frac{\partial u_l^m}{\partial x_i} + \gamma A_k\delta_{3l}\frac{\partial u_i^m}{\partial x_l} - ikA_k\delta_{1l}\frac{\partial u_i^m}{\partial x_l} - ikA_i\delta_{1l}\frac{\partial u_k^m}{\partial x_l} + \\
&+ \gamma A_i\delta_{3l}\frac{\partial u_k^m}{\partial x_l} - ikA_l\delta_{1k}\frac{\partial u_i^m}{\partial x_l} + \gamma A_l\delta_{3k}\frac{\partial u_i^m}{\partial x_l} - ikA_i\delta_{1l}\frac{\partial u_l^m}{\partial x_k} + \\
&\left. + \gamma A_i\delta_{3l}\frac{\partial u_l^m}{\partial x_k}\right) + \left(K - \frac{2}{3}\mu + B\right)\left[\delta_{ik}\left(-ikA_m\delta_{1n}\frac{\partial u_m^m}{\partial x_n} + \right.\right. \\
&\left. + \gamma A_m\delta_{3n}\frac{\partial u_m^m}{\partial x_n}\right) - ikA_l\delta_{1l}\frac{\partial u_i^m}{\partial x_k} + \gamma A_l\delta_{3l}\frac{\partial u_i^m}{\partial x_k} - ikA_i\delta_{1k}\frac{\partial u_l^m}{\partial x_l} + \\
&\left. + \gamma A_i\delta_{3k}\frac{\partial u_l^m}{\partial x_l}\right] + \frac{A}{4}\left(-ikA_k\delta_{1l}\frac{\partial u_l^m}{\partial x_i} + \gamma A_k\delta_{3l}\frac{\partial u_l^m}{\partial x_i} - ikA_l\delta_{1k}\frac{\partial u_k^m}{\partial x_l} + \right. \\
&\left. + \gamma A_l\delta_{3i}\frac{\partial u_k^m}{\partial x_l}\right) + B\left[\left(-ikA_k\delta_{1i}\frac{\partial u_l^m}{\partial x_l} + \gamma A_k\delta_{3i}\frac{\partial u_l^m}{\partial x_l} - ikA_l\delta_{1l}\frac{\partial u_k^m}{\partial x_i} + \right.\right. \\
&+ \gamma A_l\delta_{3l}\frac{\partial u_k^m}{\partial x_i} + \frac{\delta_{ik}}{2}\left(-ikA_m\delta_{1n}\frac{\partial u_n^m}{\partial x_m} + \gamma A_m\delta_{3n}\frac{\partial u_n^m}{\partial x_m} - ikA_n\delta_{1m}\frac{\partial u_m^m}{\partial x_n} + \right. \\
&\left.\left. + \gamma A_n\delta_{3m}\frac{\partial u_m^m}{\partial x_n}\right)\right] + 2C\left(-ikA_l\delta_{1l}\frac{\partial u_l^m}{\partial x_l} + \gamma A_l\delta_{3l}\frac{\partial u_l^m}{\partial x_l}\right)\delta_{ik}
\end{aligned} \tag{3.2}
$$

The selection of the system (2.2) is given as a sum of the general solution of the homogeneous system and of a particular solution of the inhomogeneous system (with consistency conditions taken into account),

$$
\begin{aligned}
&w_1 = w' e^{\varkappa_1 z} + w'' e^{\varkappa_2 z} \\
&w_3 = i\rho_1 w' e^{\varkappa_1 z} + i\rho_2 w'' e^{\varkappa_2 z} + V_1\left(\alpha_{11}\frac{\partial a_1}{\partial t} + \alpha_{12}\frac{\partial a_1}{\partial x} + \alpha_{13}\frac{\partial a_1}{\partial z} + F_{11}\right) \times \\
&\times e^{\varkappa_1 z} + V_2\left(\beta_{11}\frac{\partial a_2}{\partial t} + \beta_{12}\frac{\partial a_2}{\partial x} + \beta_{13}\frac{\partial a_2}{\partial z} + F_{12}\right) l^{\varkappa_2 z} \\
&\left(V_1 = -\frac{i}{k\varkappa_1 (c_l^2 - c_t^2)},\quad V_2 = -\frac{i}{k\varkappa_2 (c_l^2 - c_t^2)}\right)
\end{aligned} \tag{3.3}
$$

where w' and w" are constant amplitudes of the order μ^*.

For (3.3) the boundary conditions (3.1) now become

$$
\begin{aligned}
&c_t^2(\varkappa_1 + k\rho_1) w' + c_t^2(\varkappa_2 + k\rho_2) w'' = i\nu_{11}\frac{\partial a}{\partial t} + i\nu_{12}\frac{\partial a}{\partial x} + \nu_{13}\left(\frac{\partial a_1}{\partial z}\right)_0 + \\
&+ \nu_{14}\left(\frac{\partial a_2}{\partial z}\right)_0 + \Phi_{13} + \frac{c_t^2 (F_{11})_0}{k\rho_1 (c_l^2 - c_t^2)} + \frac{c_t^2 (F_{12})_0}{k\rho_2 (c_l^2 - c_t^2)} \\
&i\,[c_l^2\varkappa_1\rho_1 - k(c_l^2 - 2c_t^2)]\,w' + i\,[c_l^2\varkappa_2\rho_2 - k(c_l^2 - 2c_t^2)]\,w'' = \nu_{21}\frac{\partial a}{\partial t} + \\
&+ \nu_{22}\frac{\partial a}{\partial x} + i\nu_{23}\left(\frac{\partial a_1}{\partial z}\right)_0 + i\nu_{24}\left(\frac{\partial a_2}{\partial z}\right)_0 + \Phi_{33} - \frac{ic_l^2\,[(F_{11})_0 + (F_{12})_0]}{k(c_l^2 - c_t^2)} \\
&\left(\nu_{11} = -\frac{2c_t^3\xi\,(1 + S\rho_1\rho_2)}{\rho_1 (c_l^2 - c_t^2)},\quad \nu_{12} - \frac{2(c_l^2 - \rho_1^2 c_l^2 + \rho_1^2 c_t^2 + S\rho_1\rho_2 c_t^2)\,c_t^2}{k(c_l^2 - c_t^2)}\right.
\end{aligned} \tag{3.4}
$$

$$\nu_{14} = \frac{c_t^2\,[2c_t^2 + (\rho_2^2 - 1)(c_l^2 - c_t^2)]}{c_l^2 - c_t^2}, \quad \nu_{21} = \frac{2c_l^2 c_t \xi\,(S+1)}{c_l^2 - c_t^2}, \quad \nu_{13} = \frac{2c_t^4}{c_l^2 - c_t^2}$$

$$\nu_{22} = \frac{c_l^2\,[(c_l^2 - c_t^2)(1 - \rho_1^2) + (1 + S)(c_l^2 - c_t^2)]}{c_l^2 - c_t^2}, \quad \nu_{23} = \frac{2c_t^2 c_l^2 \rho_1}{c_l^2 - c_t^2},$$

$$\left.\nu_{24} = \frac{2c_t^2 c_l^2}{\rho_2\,(c_l^2 - c_t^2)}\right)$$

The structure of the quantities a_1 and a_2 is different in z. The determinant of the system (3.4) vanishes and its consistency condition may be regarded as shortened boundary conditions,

$$\begin{gathered} i(\nu_{11} - n\nu_{21})\frac{\partial a}{\partial t} + i(\nu_{12} - n\nu_{22})\frac{\partial a}{\partial x} + (\nu_{13} - n\nu_{23})\left(\frac{\partial a_1}{\partial z}\right)_0 + \\ + (\nu_{14} - n\nu_{24})\left(\frac{\partial a_2}{\partial z}\right)_0 + \Phi_{13} - in\Phi_{33} + \frac{(c_t^2 - n\rho_1 c_l^2)}{k(\rho_1 c_l^2 - \rho_1 c_t^2)}(F_{11})_0 + \frac{(c_t^2\rho_2 - nc_l^2)}{k\rho_2(c_l^2 - c_t^2)}(F_{12})_0 \\ \left(n = -\frac{2c_t^2\rho_1}{\rho_1 c_l^2 - c_l^2 + 2c_t^2}\right) \end{gathered} \tag{3.5}$$

where $(\partial a/\partial z)_0$ and $(\partial a_2/\partial z)_0$ are derivatives with z approaching 0.

4. Thus (2.7) together with (3.5) is a shortened formulation of the problem of the modulation of a Rayleigh surface wave by an arbitrary elastic field. The solution in the boundary layer is interesting in practice where the amplitudes of the modulated waves are considerable. By having $z \to 0$, and considering (2.7) under the condition (1.6) one is able to obtain

$$\begin{aligned} \left(\frac{\partial a_1}{\partial z}\right)_0 &= \frac{ic_t\xi}{\rho_1 c_l^2}\left[\frac{\partial a}{\partial t} + \frac{c_l^2}{c_t\xi}\frac{\partial a}{\partial x} + \frac{i(F_{11})_0 - \rho_1(F_{31})_0}{2\omega(1 - \rho_1^2)}\right] \\ \left(\frac{\partial a_2}{\partial z}\right)_0 &= \frac{i\rho_2\xi}{c_t}\left[S\frac{\partial a}{\partial t} - \frac{Sc_t}{\xi}\frac{\partial a}{\partial x} + \frac{i(F_{12})_0 - \rho_2(F_{32})_0}{2\omega(1 - \rho_2^2)}\right] \end{aligned} \tag{4.1}$$

The sought shortened equation which describes slow changes of the complex amplitude of the Rayleigh wave acted upon by the modulating field $\mathbf{u}^m$ is given by

$$\begin{gathered} \frac{\partial a}{\partial t} + v\frac{\partial a}{\partial x} = a\left(ir_1\frac{\partial u_x^m}{\partial x} + ir_2\frac{\partial u_y^m}{\partial y} + ir_3\frac{\partial u_z^m}{\partial z} + r_4\frac{\partial u_x^m}{\partial z} + r_5\frac{\partial u_z^m}{\partial x}\right) \\ \left(v = \sigma_0\left[\nu_{12} + n\nu_{22} + \frac{1}{\rho_1}(\nu_{13} - n\nu_{23}) + S\rho_2(\nu_{14} - n\nu_{24})\right]\right. \\ \left.\frac{1}{\sigma_0} = \nu_{11} + \nu_{21} + \frac{c_t\xi}{\rho_1 c_l^2}(\nu_{13} - n\nu_{23}) + \frac{\rho_2\xi S}{c_t^2}(\nu_{14} - n\nu_{24})\right) \end{gathered} \tag{4.2}$$

The expressions for the coefficients $\mathbf{r}_i$ are as follows:

$$\begin{aligned} r_1 &= \sigma_1 N_3 + \sigma_2 N_5 + \sigma_3 N_7 + \sigma_4 N_8 + \sigma_0 M_3 - n\sigma_0 M_5 \\ r_2 &= \sigma_1 N_1 + \sigma_2 N_2 + \sigma_3\rho_1 N_1 + \sigma_4\rho_2 N_2 + \sigma_0 M_1 - n\sigma_0 M_2 \\ r_3 &= \sigma_1 N_{11} + \sigma_2 N_{13} + \sigma_3 N_{15} + \sigma_4 N_{17} + \sigma_0 M_8 - n\sigma_0 M_{10} \\ r_4 &= -\sigma_1 N_4 + \sigma_2 N_6 + \sigma_3 N_8 + \sigma_4 N_9 - \sigma_0 M_4 - n\sigma_0 M_6 \\ r_5 &= -\sigma_1 N_{10} - \sigma_2 N_{12} + \sigma_3 N_{14} + \sigma_4 N_{16} - \sigma_0 M_7 - n\sigma_0 M_9 \end{aligned}$$

$$\sigma_1 = \frac{\sigma_0\,[2c_l^2(1 - \rho_1^2)(c_t^2 + n\rho_1 c_l^2) - (c_l^2 - c_t^2)(\nu_{13} - n\nu_{23})]}{2\rho_1 kc_l^2(1 - \rho_1^2)(c_l^2 - c_t^2)}$$

$$\sigma_2 = \frac{\sigma_0\,[2c_t^2(1 - \rho_2^2)(c_t^2\rho_2 + nc_l^2) - \rho_2(c_l^2 - c_t^2)(\nu_{14} - n\nu_{24})]}{2kc_t^2(1 - \rho_2^2)(c_l^2 - c_t^2)}$$

$$\sigma_3 = \frac{\sigma_0(\nu_{13} - n\nu_{23})}{kc_l^2(1 - \rho_1^2)}, \quad \sigma_4 = \frac{\sigma_0\rho_2^2(\nu_{14} - n\nu_{24})}{kc_t^2(1 - \rho_2^2)}$$

$$N_1 = -\frac{\omega^2}{c_l^2}\left(K - 2\frac{\mu}{3} + B + 2C\right), \quad N_2 = -\frac{\omega^2}{2c_t^2}(K - {}^2/_3\mu + B)$$

$$N_3 = \varkappa_1^2(2K + {}^5/_3\mu + A + 4B + 2C) - k^2(3K + 4\mu + 2A + 6B + 2C)$$

$$N_4 = \rho_1\varkappa_1^2(K + {}^4/_3\mu + {}^1/_2 A + B) - \rho_1 k^2(4K + {}^{13}/_3\mu + 2A + 4B)$$

$$N_5 = -k^2(K + {}^7/_3\mu + A + 2B)$$

$$N_6 = -k^2\rho_2(\mu + {}^1/_2A + B) - \varkappa_2 k(3K + 2\mu + A + 2B)$$

$$N_7 = -k^2\left(\mu + {}^1/_2A + B\right) + \varkappa_1^2(K + {}^{10}/_3\mu + {}^3/_2A + 3B)$$

$$N_8 = -k^2\rho_2(K + {}^4/_3\mu + {}^1/_2A + B) - k\varkappa_2(\mu + {}^1/_2A + B)$$

$$N_9 = k^2(\mu + {}^1/_2A + B) + \varkappa_1^2(K + {}^4/_3\mu + {}^1/_2A + B)$$

$$N_{10} = -k^2\rho_1(K + {}^{10}/_3\mu + {}^3/_2A + 3B) + \rho_1\varkappa_1^2(\mu + {}^1/_2A + B)$$

$$N_{11} = \varkappa_1^2(2K + {}^5/_3\mu + A + 4B + 2C) - k^2(K - {}^2/_3\mu + 2B + 2C)$$

$$N_{12} = -k^2\rho_2(K + {}^4/_3\mu + {}^1/_2A + B) - k\varkappa_2(\mu + {}^1/_2A + B)$$

$$N_{13} = k^2(\mu + {}^1/_2A + B) - k\varkappa_2(\mu + {}^1/_2A + B)$$

$$N_{14} = \varkappa_1^2(2K - {}^1/_3\mu + {}^1/_2A + 3B) - k^2(K + {}^4/_3\mu + {}^1/_2A + B)$$

$$N_{15} = \rho_1\varkappa_1^2(3K + 4\mu + 2A + 5B + 2C) - \rho_1 k^2(2K + {}^5/_3\mu + A + 4B + 2C)$$

$$N_{16} = -k^2\left(K + {}^4/_3\mu + {}^1/_2A + B\right) + \varkappa_2^2(\mu + {}^1/_2A + B)$$

$$N_{17} = \varkappa_2 k(2K + {}^5/_3\mu + A + 3B) - \rho_2 k^2(K + {}^4/_3\mu + {}^1/_2A + B)$$

$$M_1 = -\varkappa_1(K - {}^2/_3\mu + 2B) - \varkappa_2(K - {}^2/_3\mu + B + \rho_2^2 B)$$

$$M_2 = -(\varkappa_1\rho_1 + \varkappa_2\rho_2)(K - {}^2/_3\mu + 2B)$$

$$M_3 = -[\varkappa_1(K + {}^7/_3\mu + A + 2B) + \varkappa_2(K + {}^4/_3\mu + {}^1/_2A + B) + k\rho_2(\mu + {}^1/_2A + B)]$$

$$M_4 = -[\rho_1\varkappa_1(K + {}^4/_3\mu + {}^1/_2A + B) - k(K + {}^4/_3\mu + {}^1/_2A + B)]$$

$$M_5 = -[-k(K - {}^2/_3\mu + 4B + 2C) + \rho_1\varkappa_1(K - {}^2/_3\mu + 2B)]$$

$$M_6 = -\left[\varkappa_1(K + {}^4/_3\mu + {}^3/_4A + 3B) + \varkappa_2(K + {}^4/_3\mu + {}^1/_2A + B) + k\rho_2\left(\mu + {}^1/_4A + 2B\right)\right]$$

$$M_7 = -[-k(B + \mu) + \rho_1\varkappa_1(\mu + {}^1/_2A + B)]$$

$$M_8 = -[\varkappa_1(K + {}^7/_3\mu + {}^3/_4A + 2B) + k\rho_2(\mu + {}^1/_4A + B) + \varkappa_2(K + {}^1/_3\mu + {}^1/_4A + B)]$$

$$M_9 = -[\varkappa_1(K - {}^4/_3\mu + {}^3/_4A + 2B) + \varkappa_2(\mu + {}^1/_2A + B) + k\rho_2(K + {}^1/_3\mu + {}^1/_4A + B)]$$

$$M_{10} = -[\varkappa_1\rho_1(3K + 4\mu + 2A + 6B + C) + k(K + {}^1/_3\mu + {}^3/_4A + 2B + C)])$$

By adopting a complex amplitude as given by $a = a_0 e^{i\varphi}$, equations are obtained for slowly varying amplitudes and phases,

$$\frac{\partial a_0}{\partial t} + v\frac{\partial a_0}{\partial x} = a_0\left(r_4\frac{\partial u_x^m}{\partial z} + r_5\frac{\partial u_z^m}{\partial x}\right)$$
$$\frac{\partial\varphi}{\partial t} + v\frac{\partial\varphi}{\partial x} = \left(r_1\frac{\partial u_x^m}{\partial x} + r_2\frac{\partial u_y^m}{\partial y} + r_3\frac{\partial u_z^m}{\partial z}\right) \tag{4.3}$$

When analyzing the system (4.3) one notices that in contrast to the three-dimensional waves [2] a phase modulation as well as amplitude modulation occurs (of the same order of effect) when a Rayleigh wave is propagated in the presence of a modulating field $\mathbf{u}_1^m$. To be able to analyze in detail a solution of the system (4.3) some specific forms of the modulating fields will be considered such that the boundary conditions (1.2) are satisfied.

Let the modulating field be a field of uniform deformation in a rod due to the tensile force P directed along the rod axis. In this case the following relations are valid [1]:

$$\frac{\partial u_x^m}{\partial x} = \frac{P}{E}(\cos^2\theta - \sigma\sin^2\theta),\quad \frac{\partial u_y^m}{\partial y} = \frac{P}{E}(\sin^2\theta - \sigma\cos^2\theta),\quad \frac{\partial u_z^m}{\partial z} = -\frac{P\sigma}{E} \tag{4.4}$$

where E is the Young modulus, σ is the Poisson coefficient, θ is the angle between the direction of P and the x axis, the angle between the z axis and P being equal to 90°. By inserting the relations (4.4) into the system (4.3) and by integrating a solution of the form

$$\varphi = \varphi_0 + \frac{PL}{vE}[r_1(\cos^2\theta - \sigma\sin^2\theta) + r_2(\sin^2\theta - \sigma\cos^2\theta) - r_3\sigma] \tag{4.5}$$

can be obtained for the stationary state $(t = t_0 + x/v)$.

One notices when analyzing the above result that in the case of a uniform stress field there is no amplitude modulation, and for a fixed L the phase of the Rayleigh wave is proportional to the pressure P. This result is similar to the corresponding solution for three-dimensional waves [2]. If P varies in time, one can obtain a solution in a similar manner. Let $P = P_0 t$, then by using (4.4) a solution of (4.3) is obtained in the form

$$\varphi = \varphi_0 + \frac{P_0 L}{vE}[r_1(\cos^2\theta - \sigma\sin^2\theta) + r_2(\sin^2\theta - \sigma\cos^2\theta) - \sigma r_3]\left(t + \frac{L}{2v}\right) \tag{4.6}$$

In this case the phase of a Rayleigh wave changes as the square of distance; the full expression for the oscillating term in (2.1) is

$$\begin{aligned} e^{i(\omega t - kx + \varphi)} = \exp\Big\{\Big[\omega + \frac{P_0 L}{vE}(r_1\cos^2\theta - r_1\sigma\sin^2\theta + r_2\sin^2\theta - r_2\cos^2\theta\sigma - r_3\sigma)\Big]t \\ - kL - \frac{P_0 L}{2Ev}(r_1\cos^2\theta - r_1\sigma\sin^2\theta + r_2\sin^2\theta - r_2\sigma\cos^2\theta - r_3\sigma)\Big\} \end{aligned} \tag{4.7}$$

It follows from (4.7) that the frequency of a Rayleigh wave changes linearly with the path of the wave.

Let us now consider a sinusoidal **P**. Let P be a uniform standing wave in a rod, that is

$$\mathbf{P} = \mathbf{P}_0 \sin \mathbf{k}_m \mathbf{r} \cos \Omega t$$

where **r** is the radius vector, $\mathbf{k}_m$ is the wave number.

Integrating (4.3) and using (4.4) the following expression is obtained:

$$\begin{aligned} \varphi = \varphi_0 + \frac{P_0 L}{Ev}[r_1(\cos^2\theta - \sigma\sin^2\theta) + r_2(\sin^2\theta - \sigma\cos^2\theta) - r_3\sigma] \times \\ \times \left\{\frac{\sin\Delta^- L}{\Delta^- L}\cos(\Omega t - \mathbf{k}_m\mathbf{r}_\perp + \Delta^- L) + \frac{\sin\Delta^+ L}{\Delta^+ L}\cos(\Omega t + \mathbf{k}_m\mathbf{r}_\perp + \Delta^+ L)\right\} \\ \left(\Delta^- = \frac{\Omega - k_m v\cos\theta}{2v}, \quad \Delta^+ = \frac{\Omega + k_m v\cos\theta}{2v}, \quad \mathbf{k}_m\mathbf{r}_\perp = k_m y\sin\theta\right) \end{aligned} \tag{4.8}$$

It follows from (4.8) that a pure phase modulation of a Rayleigh wave takes place also in the case of three-dimensional waves. However, some differences can also be pointed out. Unlike the three-dimensional waves the excitation and reception of Rayleigh waves can take place at any point of the surface of the acoustic resonator. Therefore, in the case of modulation by standing low-frequency fields the modulation index of Rayleigh waves depends on the coordinates of the points of excitation and of reception. The integration in (4.3) does not take place over the entire length of the resonator as was the case with three-dimensional waves but between x_1 and x_2, where x_1 and x_2 are the coordinates of the points of emission and reception of the Rayleigh waves. The solution for the wave phase is given by

$$\begin{aligned} \varphi = \varphi_0 + \frac{P_0 L}{Ev}[r_1(\cos^2\theta - \sigma\sin^2\theta) + r_2(\sin^2\theta - \sigma\cos^2\theta) - r_3\sigma] \times \\ \times \Big[\frac{\sin\Delta^- L}{\Delta^- L}\cos(\Omega t + k_m x_2\cos\theta + k_m y\sin\theta + \Delta^- L) + \\ + \frac{\sin\Delta^+ L}{\Delta^+ L}\cos(\Omega t - k_m x_1\cos\theta + k_m y\sin\theta + \Delta^+ L)\Big] \end{aligned} \tag{4.9}$$

where L is the distance traveled by the wave.

Let $\mathbf{u}^m$ be the field of normal elastic waves in a plate. It is known from [3] that such waves satisfy the boundary conditions (1.2). Without loss of generality one considers the modulating field to be an antisymmetric displacement wave in the plate. The frequency of oscillations in this wave is much smaller than the frequency of the Rayleigh wave extending over the plate surface in the xy plane. In accordance with [3] the components u_i^m of the displacements are given in this case by

$$\begin{aligned} u_x^m &= -u_0^m \sin\theta \sin\varkappa z \cos(\Omega t - k_m\cos\theta x + k_m\sin\theta y) \\ u_y^m &= u_0^m \cos\theta \sin\varkappa z \cos(\Omega t - k_m\cos\theta x + k_m\sin\theta y) \\ u_z^m &= 0 \end{aligned} \tag{4.10}$$

where u_0^m is the amplitude of the modulating wave, θ is the angle between the x axis and the propagation direction of the modulating wave, $\varkappa$ is the transversal number of the modulating wave which satisfies the dispersion equation

$$\varkappa^2 + k_m^2 = \omega^2/c_t^2 \tag{4.11}$$

By inserting (4.10) into (4.3) and integrating one finds solutions of the form

$$\varphi = \varphi_0 + (r_2 - r_1)\sin\theta\cos\theta \frac{k_m u_0^m L}{v} \frac{\sin \Delta L}{\Delta L} \sin(\Omega t - k_m y \sin\theta - \Delta L) \tag{4.12}$$

where $\Delta = \Delta^-$ is given in (4.8), and one also takes into account that $\varkappa b = \pi$, b is the plate thickness. It follows from (4.12) that here pure phase modulation of the Rayleigh wave also takes place. On the whole there is no modulation for angles $\theta = 0$ and 90°. In a similar manner one can analyze modulation by using other forms of modulating fields such that the boundary conditions (1.2) are satisfied. It is noted that in a majority of cases in practice the amplitude-modulation effect described by the first equation of the system (4.3) is not attained. In special cases when the modulating wave falls on the surface $z = 0$ at an angle this effect will take place. For example, let a longitudinal elastic wave fall on the surface $z = 0$ at an angle θ to the z axis from a solid medium. It is assumed that the length of the Rayleigh wave path is smaller than the width of the front of the modulating wave and that the phase difference in the modulating wave between the emitting and the reception points of the Rayleigh waves can be ignored. In this case the total displacement $\mathbf{u}^m$ can be represented in the form [1]

$$\mathbf{u}^m = (u_0^m \mathbf{n}_0 e^{i\mathbf{k}_0\mathbf{r}} + u_l^m \mathbf{n}_l e^{i\mathbf{k}_l\mathbf{r}} + u_t^m [\mathbf{a}\mathbf{n}_t] e^{i\mathbf{k}_t\mathbf{r}}) e^{i\Omega t} \tag{4.13}$$

where $\mathbf{n}_0$, $\mathbf{n}_e$, and $\mathbf{n}_t$ are the unit vectors in the direction of the incoming longitudinal wave, the reflected longitudinal and the reflected displacement wave respectively, u_0^m, u_l^m, and u_t^m are the amplitudes of the corresponding displacements, and $\mathbf{k}_0$, $\mathbf{k}_l$, and $\mathbf{k}_t$ are the wave vectors, $\mathbf{a}$ is the unit vector in the direction of z. The absolute values of the wave vectors are given by $k_0 = k_l = \Omega/c_l$, $k_t = \Omega/c_t$, and the angles θ_0, θ_l, and θ_t are related by

$$\theta_0 = \theta_l, \quad \sin\theta_t = \sin(\theta_0 c_t/c_l)$$

Using (4.13) one can write the expression for $\partial u_z^m/\partial x$ as

$$\partial u_z^m/\partial x = [k_0 (u_0^m - u_l^m) \sin\theta_0\cos\theta_l + {}^1/_2 u_t^m k_t (\cos^2\theta_t - \sin^2\theta_t)] \sin\Omega t \tag{4.14}$$

where the amplitudes u^m are given by the expressions

$$u_l^m = u_0^m \frac{c_t^2 \sin 2\theta_t \sin 2\theta_0 - c_l^2 \cos^2 2\theta_t}{c_t^2 \sin 2\theta_t \sin 2\theta_0 + c_l^2 \cos^2 2\theta_t}, \qquad u_t^m = u_0^m \frac{2c_l c_t \sin 2\theta_0 \cos 2\theta_t}{c_t^2 \sin 2\theta_t \sin 2\theta_0 + c_l^2 \cos^2 \theta_t}$$

By inserting (4.14) in the first of Eqs. (4.3) and integrating, one obtains a solution in the form

$$\varphi = \varphi_0 + [k_0 (u_0^m + u_l^m)\cos^2\theta_0 + u_t^m k_t \cos^2\theta_t \sin\theta_t] \frac{r_3 L}{v} \frac{\sin \Omega L/2v}{\Omega L/2v} \sin\left(\Omega t - \frac{\Omega L}{v}\right) \tag{4.15}$$

One obtains the expression

$$\partial u_z^m/\partial z = [k_0 (u_0^m + u_l^m)\cos^2\theta_0 + u_t^2 k_t \cos^2\theta_t \sin\theta_t] \sin\Omega t \tag{4.16}$$

for $\partial u_z^m/\partial z$ of (4.13).

The solution for the phase is given by

$$a_0 = a_0^* \exp\left\{\left[k_0 (u_0^m - u_l^m) \sin\theta_0\cos\theta_0 + u_t^m k_t \frac{1}{2}(\cos^2\theta_t - \sin^2\theta_t)\right] \frac{r_5 L}{v} \frac{\sin \Omega L/2v}{\Omega L/2v} \sin\left(\Omega t - \frac{\Omega L}{v}\right)\right\} \tag{4.17}$$

In accordance with (4.15) and (4.17), both phase and amplitude modulation take place when a Rayleigh wave is propagated in the field of incident longitudinal waves, of reflected longitudinal and of transversal waves. The depth of the amplitude modulation and the index of the phase modulation depend in different ways on the amplitudes of the modulating waves and their angular relations.

It is noted that the effect of nonresonance parametric interaction between surface waves and interior elastic fields in a solid medium can be employed for parametric display and for estimating the intensity of these fields.

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